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Problem: Let \( n \) be a positive integer, and let \( f \) be a continuous real-valued function on \([0,1]\) with the property that \( \int_0^1 x^k f(x) \, dx = 1 \) for \( 0 \leq k \leq n - 1 \). Prove that \( \int_0^1 (f(x))^2 \, dx \geq n^2 \).
(The case \( n = 2 \) appeared in the 55th Romanian Mathematical Olympiad, Gazeta Mathematica 5 (2004), 219.)

Solution: The polynomials \( Q_k(x) = \sqrt{2} P_k(2x - 1) \), where \( P_k(x) = \sqrt{\frac{2k+1}{2} \frac{1}{2^{k!}} \frac{d^k}{dx^k} (x^2 - 1)^k} \) are Legendre polynomials [1], are orthonormal in \( L^2[0,1] \). So, by Bessel inequality [1] and using the hypothesis, we have
\[
\sum_{k=0}^{n-1} |Q_k(1)|^2 = \sum_{k=0}^{n-1} |< f, Q_k >|^2 \leq \| f \|_2^2 = \int_0^1 (f(x))^2 \, dx.
\]
But, \( Q_k(1) = \sqrt{2} P_k(1) = \sqrt{2k+1} \), and so \( \int_0^1 (f(x))^2 \, dx \geq \sum_{k=0}^{n-1} (2k + 1) = 2 \sum_{k=1}^{n-1} k + n = n^2 \).

REFERENCES