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Problem: Let ABC be a triangle with sides a, b and c, all different, and corresponding angles α, β, and γ. Show that

$$(\mathbf{a}) \qquad (a+b)\cot\big(\beta+\frac{1}{2}\gamma\big)+(b+c)\cot\big(\gamma+\frac{1}{2}\alpha\big)+(c+a)\cot\big(\alpha+\frac{1}{2}\beta\big)=0.$$

$$(\mathbf{b}) \qquad (a-b)\tan\big(\beta+\frac{1}{2}\gamma\big)+(b-c)\tan\big(\gamma+\frac{1}{2}\alpha\big)+(c-a)\tan\big(\alpha+\frac{1}{2}\beta\big)=4(R+r),$$

where R is the circumradius of the the triangle and r the inradius.

Solution (a): The left hand side of (**a**) is equal to

$$\begin{aligned}
& 2R\left[(\sin\alpha+\sin\beta)\tan\frac{\alpha-\beta}{2}+(\sin\beta+\sin\gamma)\tan\frac{\beta-\gamma}{2}+(\sin\gamma+\sin\alpha)\tan\frac{\gamma-\alpha}{2}\right] \\
= \; & 2R\left[2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}+2\sin\frac{\beta+\gamma}{2}\sin\frac{\beta-\gamma}{2}+2\sin\frac{\gamma+\alpha}{2}\sin\frac{\gamma-\alpha}{2}\right] \\
= \; & 2R\left[\cos\beta-\cos\alpha+\cos\gamma-\cos\beta+\cos\alpha-\cos\gamma\right]=0.
\end{aligned}$$

(b): Since, $r=4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$, the left hand side of (**b**) is

$$\begin{aligned}
& 2R\left[(\sin\alpha-\sin\beta)\cot(\frac{\alpha-\beta}{2})+(\sin\beta-\sin\gamma)\cot(\frac{\beta-\gamma}{2})+(\sin\gamma-\sin\alpha)\cot(\frac{\gamma-\alpha}{2})\right] \\
= \; & 2R\left[2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2\cos\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}+2\cos\frac{\gamma+\alpha}{2}\cos\frac{\gamma-\alpha}{2}\right] \\
= \; & 4R\left[\cos\alpha+\cos\beta+\cos\gamma\right]=4R\left[2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+1-2\sin^2\frac{\gamma}{2}\right] \\
= \; & 4R\left(1+2\sin\frac{\gamma}{2}\left[\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}\right]\right)=4R\left(1+4\sin\frac{\gamma}{2}\sin\frac{\beta}{2}\sin\frac{\alpha}{2}\right)=4(R+r).
\end{aligned}$$