Hilbert Series of Monomial Algebras

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Abstract

Let P be the polynomial ring $K[x_1, ..., x_n]$ over a field K and let I be a monomial ideal of P. In this paper explicit formulae for the Hilbert series and Hilbert function of P/I (with standard grading) is computed in terms of the generators of I.

Hilbert series and Hilbert function are important invariants of graded rings and modules and they contain considerable information. Any affine K-Algebra is of the form $K[x_1, ..., x_n]/I$ for some positive integer n and some ideal I. It is well known that the Hilbert series of P/I is equal to the Hilbert series of $P/\mathrm{In}(I)$ where $\mathrm{In}(I)$ is the initial ideal of I with respect to some term ordering. The ideal $\mathrm{In}(I)$ is a monomial ideal and it's generators can be obtained for example from a Gröbner basis for I. In this paper, for a monomial ideal I of P, which is generated by monomials $t_1, ...t_r$, we will find explicitly the Hilbert series and Hilbert function of P/I in term of degrees of the t_i's.

Lemma 1. Let K be a field and $P = K[x_1, ..., x_n]$ with standard grading. Let $f \in P\backslash\{0\}$ be a homogeneous polynomial of degree d. Hilbert function and Hilbert series of the graded $P-$module $P/\langle f\rangle$ are of the forms

$$HF(P/\langle f\rangle, i) = HF(P, i) - HF(P, i - d),$$

$$HS(P/\langle f\rangle, z) = \frac{1 - z^d}{(1 - z)^n}.$$

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Proof. Consider the exact sequence

$$0 \longrightarrow P(-d) \xrightarrow{f} P \longrightarrow P/\langle f\rangle \longrightarrow 0. \tag{1}$$

Then,

$$HF(P/\langle f\rangle, i) = HF(P,i) - HF(P(-d),i) = HF(P,i) - HF(P,i-d)$$

and

$$HS(P/\langle f\rangle, z) = HS(P,z) - HS(P(-d),z) = \frac{1-z^d}{(1-z)^n}. \qquad \square$$

Let M be a finitely generated graded $P-$module and let $f \in P\backslash\{0\}$ be a homogeneous polynomial of degree d. Note that multiplication by f defines a homogeneous $P-$ linear map $M(-d) \longrightarrow M$ whose kernel is $[(0:_M \langle f\rangle)](-d)$ and whose cokernel is M/fM. Therefore we have the exact sequence

$$0 \longrightarrow \left[\frac{M}{(0:_M \langle f\rangle)}\right](-d) \xrightarrow{f} M \longrightarrow M/fM \longrightarrow 0.$$

Now consider the case $M = P/I$, where I is a homogeneous ideal. The homogenous exact sequence

$$0 \longrightarrow \left[\frac{P}{(I:_{P/I} \langle f\rangle)}\right](-d) \xrightarrow{f} P/I \longrightarrow P/(I+\langle f\rangle) \longrightarrow 0$$

can be obtained, because $(0:_{P/I}\langle f\rangle) = (I:_P \langle f\rangle)/I$. Let $I = \langle t_1\rangle$, where t_1 is a monomial and $deg(t_1) = d_1$, and let t_2 be a monomial of degree d_2. The above exact sequence can be rewritten as

$$0 \longrightarrow \left[\frac{P}{(\langle t_1\rangle :_P \langle t_2\rangle)}\right](-d_2) \xrightarrow{t_2} P/I \longrightarrow P/\langle t_1, t_2\rangle \longrightarrow 0.$$

We denote degree of t_i by $|t_i|$ and degree of $lcm(t_1,t_2)$ by $|t_1 \vee t_2|$. Using the Lemma 1 and the exact sequence (1) and the fact that $(\langle t_1\rangle :_P \langle t_2\rangle) = (\frac{1}{t_2}lcm(t_1,t_2))$, we can compute the Hilbert function and Hilbert series of $P/\langle t_1,t_2\rangle$.

$$\begin{aligned} HF\left(P/\langle t_1,t_2\rangle, i\right) &= HF\left(P/\langle t_1\rangle, i\right) - HF\left(\left[\frac{P}{(\langle t_1\rangle :_{P/I} \langle t_2\rangle)}\right](-|t_2|), i\right) \\ &= HF(P,i) - HF(P, i-|t_1|) - HF(P, i-|t_1|) \\ &+ HF(P, i-|t_1 \vee t_2|) \end{aligned}$$

and

$$\begin{aligned} HS(P/\langle t_1,t_2\rangle,z) &= HS(P/\langle t_1\rangle,z) - HS\left(\left[\frac{P}{(\langle t_1\rangle :_P \langle t_2\rangle)}\right](-|t_2|),z\right) \\ &= \frac{1-z^{|t_1|}}{(1-z)^n} - \frac{z^{|t_2|}(1-z^{|t_1\vee t_2|-|t_2|})}{(1-z)^n} \\ &= \frac{1-z^{|t_1|}-z^{|t_2|}+z^{|t_1\vee t_2|}}{(1-z)^n} \end{aligned}$$

In the case of $I=\langle t_1,t_2\rangle$, we have the exact sequence

$$0\longrightarrow \left[\frac{P}{(\langle t_1,t_2\rangle :_P \langle t_3\rangle)}\right](-|t_3|) \xrightarrow{t_3} P/I \longrightarrow P/\langle t_1,t_2,t_3\rangle \longrightarrow 0,$$

and

$$(\langle t_1,t_2\rangle :_P \langle t_3\rangle) = (\frac{1}{t_3}(\langle t_1\rangle\cap\langle t_3\rangle), \frac{1}{t_3}(\langle t_2\rangle\cap\langle t_3\rangle) = (\frac{1}{t_3}lcm(t_1,t_3), \frac{1}{t_3}lcm(t_2,t_3))$$

also we have

$$lcm(lcm(t_1,t_3), lcm(t_2,t_3)) = lcm(t_1,t_2,t_3).$$

Then

$$\begin{aligned} HF(P/\langle t_1,t_2,t_3\rangle,i) &= HF(P,i) - HF(P,i-|t_1|) - HF(P,i-|t_2|) \\ &\quad -HF(P,i-|t_3|) + HF(P,i-|t_1\vee t_2|) \\ &\quad +HF(P,i-|t_1\vee t_3|) + HF(P,i-|t_2\vee t_3|) \\ &\quad -HF(P,i-|t_1\vee t_2\vee t_3|), \end{aligned}$$

and

$$HS(P/\langle t_1,t_2,t_3\rangle,z) = \frac{1-z^{|t_1|}-z^{|t_2|}-z^{|t_3|}+z^{|t_1\vee t_2|}+z^{|t_1\vee t_3|}+z^{|t_2\vee t_3|}-z^{|t_1\vee t_2\vee t_3|}}{(1-z)^n}.$$

In the general case if $I=\langle t_1,...,t_{r-1}\rangle$ where $t_1,...,t_{r-1}$ and t_r are nonzero monomials, we have the exact sequence

$$0\longrightarrow \left[\frac{P}{(\langle t_1,...,t_{r-1}\rangle :_P \langle t_r\rangle)}\right](-|t_r|) \xrightarrow{t_r} P/I \longrightarrow P/\langle t_1,...,t_{r-1},t_r\rangle \longrightarrow 0$$

and $(\langle t_1,...,t_{r-1}\rangle :_P \langle t_r\rangle) = \langle g_{r1},...,g_{rr-1}\rangle$, where $g_{sj}=\frac{lcm(g_s,g_j)}{g_s}$ for $j=1,...,s-1$. So, we have proven the following theorem.

Theorem 1. Let K be a field. Let $P = K[x_1, ..., x_n]$ with standard grading and $t_1, ..., t_r$ be nonzero monomials in P, and $deg(t_i) = |t_i|$. The Hilbert function and Hilbert series of $P/\langle t_1, \ldots, t_r\rangle$ are as following.

$$\begin{aligned} HF(P/\langle t_1, ..., t_r\rangle, i) &= HF(P, i) - \sum_{j=1}^{r} HF(P, i - |t_j|) \\ &+ \sum_{1\leq j_1<j_2\leq r} HF(P, i - |t_{j_1} \vee t_{j_2}|) \\ &- \sum_{1\leq j_1<j_2<j_3\leq r} HF(P, i - |t_{j_1} \vee t_{j_2} \vee t_{j_3}|) \\ &+ ... + (-1)^s \sum_{1\leq j_1<j_2<...<j_s\leq r} HF(P, i - |t_{j_1} \vee t_{j_2} \vee ... \vee t_{j_s}|) \\ &+ ... + (-1)^r HF(P, i - |t_1 \vee t_2 \vee ... \vee t_r|) \end{aligned}$$

and

$$HS(P/\langle t_1, ..., t_r\rangle, z) = \frac{1 - \sum_{j=1}^{r} z^{|t_j|} + ... + (-1)^s \sum_{1\leq j_1<...<j_s\leq r} z^{|t_{j_1}\vee t_{j_2}\vee ...\vee t_{j_s}|} + ... + (-1)^r z^{|t_1\vee t_2\vee ...\vee t_r|}}{(1 - z)^n}.$$

Corollary 1. Let $I = \langle t_1, ..., t_r\rangle$ such that $t_1, ..., t_r$ be monomials in P and $deg(t_i) = d_i$. If t_i's are pairwise coprime, then

$$HS(P/\langle t_1, ..., t_r\rangle, z) = \frac{\prod_{i=1}^{r}(1 - z^{d_i})}{(1 - z)^n} \qquad \square$$

Corollary 2. Let $P = K[x_1, ..., x_n]$ be graded by a matrix $W = (w_{ij}) \in Mat_{m,n}(\mathbb{Z})$, then,

$$\begin{aligned} &HS_W(P/\langle t_1, ..., t_r\rangle, z_1, ..., z_m) = \\ &\frac{1 - \sum_{i=1}^{r} z_1^{|t_i|_{w_1}} ... z_m^{|t_i|_{w_m}} + \sum_{1\leq i_1<i_2\leq r} z_1^{|t_{i_1}\vee t_{i_2}|_{w_1}} ... z_m^{|t_{i_1}\vee t_{i_2}|_{w_m}}}{\prod_{j=1}^{n}(1 - z_1^{w_{1j}} ... z_m^{w_{mj}})} \\ &+ ... + \frac{(-1)^n \sum_{1\leq i_1<i_2<...<i_r\leq r} z_1^{|t_{i_1}\vee t_{i_2}\vee ...\vee t_{i_r}|_{w_1}} ... z_m^{|t_{i_1}\vee t_{i_2}\vee ...\vee t_{i_r}|_{w_m}}}{\prod_{j=1}^{n}(1 - z_1^{w_{1j}} ... z_m^{w_{mj}})} \end{aligned}$$

where $deg_W(x_i) = (w_{1i}, ..., w_{mi})$ and $deg_W(t_i) = (|t_i|_{w_1}, ..., |t_i|_{w_m})$.

Example 1. Let $P = \mathbb{Q}[x_1, x_2, x_3]$ with standard grading and

$$I = \langle x_1^3 x_2, x_2 x_3^2, x_2^2 x_3, x_3^4 \rangle.$$

The Hilbert series of P/I is

$$\begin{aligned} HS(P/\langle f_1, f_2, f_3, f_4 \rangle, z) &= \frac{1 - 2z^3 - z^4 + z^5 + 2z^6 - z^7}{(1-z)^3} \\ &= \frac{1 + 2z + 3z^2 + 2z^3 - z^5}{(1-z)}. \end{aligned}$$

Example 2. Let $P = Q[x_1, x_2, x_3]$ be graded by $W = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ and $I =< x_1^3 x_2, x_2 x_3^2, x_2^2 x_3, x_3^4 >$. The multigraded Hilbert Series of P/I is

$$HS_W(P/I, z_1, z_2) = \frac{-z_1^5 z_2^{-1} + z_1^3 z_2^{-3} + z_1^2 z_2^{-2} + z_1^3 + z_1^2 z_2^{-1} + z_1^2 + -z_1 z_2^{-1} + z_1 + 1}{1 - z_1}.$$

Note that in the standard form of Hilbert series, the nominator is not divisible by $1 - z$, and in this situation, the h-vector of the graded ring, $(h_0, \ldots, h_d)$, is defined by coefficients of the nominator. To find the standard form of Hilbert series, suppose the nominator of the formula in Theorem 1 is $Q(z)$. To find and cancel all factors of $(1 - z)$ in $Q(z)$ we can write Taylor expansion of $Q(z)$ at the point $z = 1$:

$$Q(z) = Q(1) - Q'(1)(1 - z) + \frac{Q''(1)}{2!}(1 - z)^2 + \cdots,$$

therefore, if there is the factor $(1 - z)^{n-d}$ in the nominator, then,

$$Q(1) = \cdots = Q^{(n-d-1)}(1) = 0$$

and we get

$$H(z) = (-1)^{n-d} \frac{Q^{(n-d)}(1)}{(n-d)!} + (-1)^{n-d+1} \frac{Q^{(n-d+1)}(1)}{(n-d+1)!} z + \cdots.$$

Let $b_0, b_1, \ldots, b_d$ be defined as follows:

$$\begin{aligned} b_i &= (-1)\sum_{j=1}^{r}\binom{|t_j|}{n-d+i} + \cdots \\ &+(-1)^s \sum_{1\le j_1<\cdots<j_s\le r}\binom{|t_{j_1}\vee\cdots\vee t_{j_s}|}{n-d+i} + \cdots (-1)^r\binom{|t_1\vee\cdots\vee t_r|}{n-d+i} \end{aligned}$$

Then we have the following theorem.

Theorem 2. With the assumptions of Theorem 1 and notations above, the h-vector of $P/\langle t_1, \ldots, t_r\rangle$ is $(h_0, h_1, \ldots, h_d)$, where

$$h_i = \sum_{j=0}^{d}(-1)^{d+i+j}\binom{j}{i}b_j.$$

and d is height of the ideal. In particular, $h_0 = 1$ can be a criteria for finding d.

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