A Characterization of Uniquely Vertex Colorable Graphs Using Minimal Defining Sets

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Abstract

A *defining set* (*of vertex coloring*) of a graph G is a set of vertices S with an assignment of colors to its elements which has a unique completion to a proper coloring of G. We define a *minimal defining set* to be a defining set which does not properly contain another defining set. If G is a uniquely vertex colorable graph, clearly its minimum defining sets are of size $\chi(G)-1$. It is shown that for a coloring of G, if all minimal defining sets of G are of size $\chi(G)-1$, then G is a uniquely vertex colorable graph.

1 Introduction and Preliminaries

We follow [2] for terminologies and notations not defined here and we consider finite undirected simple graphs. By a coloring, we really mean a $\chi(G)$- coloring and always use $\{1,\ldots,\chi(G)\}$ as the set of colors. For a given coloring c we denote $c^{-1}(i)$ by C_i. Two colorings c and c' are called *isomorphic* if we can obtain c' from c by permuting the colors. A graph G is said to be *uniquely vertex colorable*, or UVC for short, if all $\chi(G)$–colorings of G are isomorphic. Uniquely vertex colorable graphs have been widely studied. For some references see [1], [3], [4], and [7].

Here we introduce some concepts which play key roles in the proof of our theorem. For every pair of colorings c and c' of G, we define the $\chi(G)\times\chi(G)$ *overlapping matrix* as $\Omega(c,c')=[\omega_{ij}]$, where $\omega_{ij}=|C_i\cap C'_j|$. The following lemma is a straightforward consequence of the celebrated König's lemma [8] which we mention it without the proof:

Lemma 1.1 *For any two colorings c and c' of a graph G, there is a coloring c'' isomorphic to c' such that all diagonal entries of $\Omega(c,c'')$ are non-zero.*

Note that for two isomorphic colorings c and c', $\Omega(c,c')$ is similar to a diagonal matrix. Also, for a UVC graph, all overlapping matrices are similar. By the *overlapping vector* of c and c' we mean the vector $\omega(c,c') = (\omega_{11}, \ldots, \omega_{\chi-1,\chi-1})$. For a given coloring c we consider the set X_c of all vectors $\omega(c,c')$, where c' is any coloring non-isomorphic to c, and $\omega(c,c')$ has no zero entry. Let $\prec$ denotes the lexicographical ordering on X_c. If G is not UVC then by Lemma 1.1, X_c is non-empty and $(X_c, \prec)$ has a maximal element, say $\omega(c,c^*)$. We call such c^*, a *maximal overlap coloring* of c.

The concept of a defining set is studied, to some extent, for block designs and also under different name, a critical set, for latin squares. Mahmoodian [5] introduced this concept for vertex colorings of graphs. A *defining set* (*of vertex coloring*) of a graph G is a set of vertices S with an assignment of colors to its elements which has a unique extension to a proper coloring of G. In [6] the *minimum defining sets*, namely defining sets having the minimum number of vertices are studied. The size of such a defining set is denoted by $d_v(G)$, or briefly by d_v. For an example see Figure 1. We define a *minimal defining set* to be a defining set which does not properly contain another defining set. We denote by $D_v(G)$ the cardinality of a minimal defining set which has maximum cardinality among all such sets.

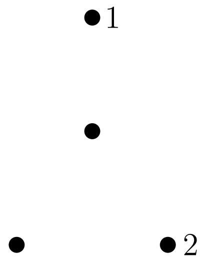

Figure 1. A graph with $d_v = \chi - 1$

Let c be a coloring of G. A *defining set of* c is a defining set of G which is uniquely extendable to c. The concepts of minimum and minimal defining sets of c can be defined analogously.

The following results are immediate consequences of the above definitions:

(a) At least $\chi - 1$ different colors appear in any defining set.

(b) $d_v \geq \chi - 1$.

(c) In a given coloring c, if one of colors other than $c(v)$, does not appear in the neighborhood of v, then every defining set of c contains v.

(d) If $\deg(v) < \chi - 1$, then every defining set contains v.

2 Unique Colorings and Defining Sets

We study the relationship between unique colorings and defining sets. If G is UVC then $d_v(G) = \chi(G) - 1$, but the converse is not necessarily true. For example see Figure 1. However, if all the minimal defining sets of G are of size $\chi(G) - 1$, i.e. $D_v(G) = d_v(G) = \chi(G) - 1$, then we show that G is a uniquely vertex colorable graph. We start by the following lemma:

Lemma 2.1 *Let G be a graph and $c : V(G) \to \{1, \ldots, \chi(G)\}$ be a coloring of G, in which all minimal defining sets are of size $\chi(G) - 1$. Then for every $u \in V(G)$, all colors other than $c(u)$ appear in $N(u)$.*

Proof. We write k for $\chi(G)$, for short. Suppose that there is a vertex for which the number of colors appearing in its neighborhood is less than $k - 1$. There are two possibilities:

Case 1. There is a vertex v with color i such that a color j does not appear in $N(v)$, but any vertex in C_j has at least a neighbor from other color classes. Clearly, $V(G) \setminus C_j$ is a defining set of c, and therefore contains a minimal defining set, say S, of size $k - 1$. This set intersects any color class in at most one vertex, and it contains the vertex v. Now, give the color i to the vertices of C_j and the color j to the vertices of $C_i \setminus v$. What we obtain is another extension of S to a coloring of G which is a contradiction.

Case 2. There is a sequence of vertices, $v_1, \ldots, v_l$, where $v_i \in C_i$, $1 \leq i \leq l$, such that the color $(i + 1) \pmod{l}$ does not appear in $N(v_i)$. Every defining set S of c contains $v_i, 1 \leq i \leq l$, Let S be a minimal defining set of size $k - 1$. Note that v_i, $1 \leq i \leq l$ is the only vertex of color i in S. In the following we show that there is another extension of S to a coloring of G. Apply the permutation $(l, l - 1, \ldots, 1)$ on the colors of vertices of $\cup_{i=1}^{l} C_i \setminus \{v_1, \ldots, v_l\}$. The graph $H = G[\cup_{i=1}^{l} C_i]$ has chromatic number l. If $|C_i| = 1$ for all i then H must be K_l which contradicts the choice of v_i's. Thus $|C_i| \geq 2$ for some i, which implies that our recoloring actually yields another extension of S, which is a contradiction. □

Now, we are ready to present our main result. As mentioned in Section 1, the concept of maximal overlap colorings has a key role in our proof.

Theorem 2.2 *Let G be a graph and $c : V(G) \to \{1, \ldots, \chi(G)\}$ be a coloring of G. All minimal defining sets of coloring c are of size $\chi(G) - 1$ iff G is a uniquely (vertex) colorable graph.*

Proof. Write k for $\chi(G)$, for short. If G is UVC then obviously all minimal defining sets are of size $k - 1$. Now suppose G is not UVC. Let c^* be a maximal overlap coloring of c. Set $A_{ij} = C_i^* \cap C_j$ and $D = \cup_{i=1}^{k} A_{ii}$. Note that $C_i = \cup_{j=1}^{k} A_{ji}$ and $C_i^* = \cup_{j=1}^{k} A_{ij}$. Since c and c^* are two non-isomorphic colorings of G, if we show that A_{ij} is empty for $1 \leq i \neq j \leq k$, then we will have a contradiction from which the result follows.

First we show that for each i, $1 \leq i \leq k - 1$, we have $A_{ki} = \emptyset$. Suppose on the contrary, $v \in A_{ki}$ for some $1 \leq i \leq k-1$. We claim that the set $S = (D \setminus A_{kk}) \cup \{v\}$ is a defining set of c. Otherwise, there is another coloring c' non-isomorphic to c, such that c' is an extension of S and $\omega(c, c^*) \prec \omega(c, c')$, which contradicts the choice of c^*. Now, S must contain a minimal defining set of c, say S'. By our hypothesis, $|S'| = k - 1$. Next, we estimate the size of S' in another way. Obviously $v \in S'$. If for some j, $S' \cap A_{jj} = \emptyset$, where $1 \leq j \leq k - 1$ and $j \neq i$, then at most $k - 2$ colors appear in S' which is not possible. Also, if $S' \cap A_{ii} = \emptyset$, then by giving the color k to the vertices of C_i^* and giving the color i to the vertices of C_k^*, we get a new coloring non-isomorphic to c which is another extension of S. All these imply that $|S'| \geq k$. This contradiction shows that $A_{ki} = \emptyset$, for all $1 \leq i \leq k - 1$.

Now, we show that $A_{k-1,i}$ is empty for $1 \leq i \leq k - 2$. Suppose $v \in A_{k-1,i}$ for some $1 \leq i \leq k - 2$. Put $S = (D \setminus A_{k-1,k-1}) \cup \{v\}$. Note that if $A_{kk} = \emptyset$, then as in previous paragraph, $C_k^* = \emptyset$, which implies that G can be colored by fewer than k colors. Hence, $A_{kk} \neq \emptyset$ and therefore in S, precisely $k - 1$ colors appear. Again, by the choice of c^* we can prove that S is a defining set of c, and so it must contain a minimal defining set, say S', with $|S'| = k - 1$. Similar to the above, $v \in S'$ and $S' \cap A_{jj} \neq \emptyset$ for all $j \neq k - 1$. But this follows $|S'| \geq k$, which is again a contradiction. A similar argument can be used to prove that for all $i > j$, A_{ij} is empty.

To complete the proof, we show that A_{ij} is also empty for $i < j$. To do this we use Lemma 2.1 as follows. By what was proved above, we have $C_1 = A_{11}$ and so on, if $v \in A_{1j}$ for some $2 \leq j \leq k$, then v has no neighbor in C_1 which contradicts Lemma 2.1. Also, if $v \in A_{2j}$, for some $3 \leq j \leq k$, then by noting that $C_2 = \cup_{i=1}^{k} A_{i2} = A_{22}$, v can not have any neighbor in C_2. Continuing this argument we get $A_{ij} = \emptyset$, for $i < j$, as desired. □

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