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Greedy defining sets in graphs and Latin squares

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Abstract

Greedy algorithm sometimes uses more than $\chi(G)$ colors while coloring a graph G. A greedy defining set is an object to eliminate these extra colors so that the greedy coloring results in a minimum coloring of an order graph G. In this note we report some of the previous results as well as new results on greedy defining sets of graphs and Latin squares.

Keywords: Graph coloring, Greedy coloring, Defining set, Latin square.

1 Greedy defining sets in graphs

Let G be a graph and σ an order on the vertex set of G which orders the vertices as $v_1 \leq v_2 \leq \ldots \leq v_n$. Let's consider the greedy algorithm associated to the order σ to color the vertices of G with natural numbers $\{1, 2, \ldots\}$. If at some vertex the algorithm have to use a color larger than $\chi(G)$ then we say that the algorithm fails. If we want the greedy algorithm to succeed, then we need to pre-color some of the vertices in G before the algorithm is invoked. So we define a Greedy Defining Set (GDS) to be a subset of vertices in G together

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with a pre-coloring of S, that will cause the greedy algorithm to successfully color the whole graph G with $\chi(G)$ colors. It is understood that the algorithm skips over the vertices that are part of the defining set. Greedy defining sets of graphs were first defined and studied by the author in [4]. This concept has a close relationship with the concept of defining set in vertex coloring of graphs which is widely studied in the literature, see [2] for the first paper on this subject and [1] for a recent survey. In the sequel the formal definition follows.

Definition 1.1 For a graph G and an order σ on $V(G)$, a *greedy defining set* is a subset S of $V(G)$ with an assignment of colors to vertices in S, such that the pre-coloring can be extended to a $\chi(G)$- coloring of G by the greedy coloring of (G, σ) and fixing the color s of S. The *greedy defining number* of G is the size of a greedy defining set which has minimum cardinality, and is denoted by $\mathrm{GDN}(G, \sigma)$. A greedy defining set for a $\chi(G)$-coloring C of G is a greedy defining set of G which results in C. The size of a greedy defining set of C with the smallest cardinality is denoted by $\mathrm{GDN}(G, \sigma, C)$.

Given an order graph (G, σ) and C a $\chi(G)$-coloring of G. Consider a subset of vertices S together with the coloring of S obtained by restricting C on S. In [4] a necessary and sufficient condition for S to be a GDS has been obtained. Also in [4], the computational complexity of determining the greedy defining number of a coloring of an ordered graph has been studied.

Theorem 1.2 ([4]) *The following problem is $\mathcal{NP}$-complete:*
Instance: *An ordered graph (G, σ), a $\chi(G)$-coloring C and integer k.*
Question: $\mathrm{GDN}(G, \sigma, C) \leq k$*?*

It is asked in [4] that given an ordered graph (G, σ), whether to determine $GDN(G, \sigma)$ is an $\mathcal{NP}$-complete problem? This problem is in fact the uncolored version of the above theorem where no coloring of graph is given as a part of input. In this note we answer this problem affirmatively.
By VERTEX COVER we mean the problem which for a given graph H and an integer k asks whether G contain a vertex cover of at most k vertices. Using a reduction from VERTEX COVER we could prove the following theorem.

Theorem 1.3 *Given an ordered bipartite graph G and a positive integer k. It is $\mathcal{NP}$-complete to determine whether $GDN(G) \leq k$.*

On the other hand we show that the problem has a polynomial time solution for forests.

Theorem 1.4 *There exists an efficient algorithm to determine the greedy* *defining number of a forest.*

2 Latin squares

An $n \times n$ Latin square L is equivalent to a vertex n-coloring of the Cartesian product $K_n \square K_n$. A greedy defining set for L is a greedy defining set for $K_n \square K_n$ which results in the coloring of $K_n \square K_n$ which is equivalent to L. Greedy defining sets in Latin squares were studied first time in [5]. For additional results see [3]. Let g_n stand for the smallest size of a greedy defining set among all Latin squares of order n. We have the following proposition.

Proposition 2.1 *We have $g_n = 0$ if and only if n is a power of 2 and the Latin square so defined is $L_2 \oplus L_2 \oplus \cdots \oplus L_2$ where L_2 is the Latin square of order 2 with the first row in natural order and $\oplus$ is the direct product operator.*

Regarding to greedy defining sets of Latin squares we specially interest in studying g_n. We first give a recursive result.

Theorem 2.2 *Suppose $n = rs$, then*

$$g_n \leq r^2 g_s + (s^2 - g_s) g_r.$$

Our second result whose proof is based on a recursive method follows.

Theorem 2.3 *Suppose $n = 2^k - 1$ for some integer $k > 1$, then*

$$g_n \leq n - \log(n + 1).$$

Using Theorem 2.2 and 2.3 we obtain the following corollary which shows that g_n grows at most linearly for some infinite subfamilies of natural numbers.

Corollary 2.4 *Suppose $n = 2^k - 2^t$, where k is an arbitrary positive integer and t an arbitrary fixed integer with $0 \leq t \leq k$. Setting $\lambda = 2^t$, we have*

$$g_n \leq \lambda n - \lambda^2 (k - t).$$

We can now mention our main conjecture.

Conjecture 2.5 $g_n = \mathcal{O}(n)$.

Finally we pose our second problem which concerns the complexity of determining the greedy defining number of Latin squares. We saw already that determining the minimum greedy defining number of an ordered bipartite graph is $\mathcal{NP}$-complete problem.

Problem 2.6 *Is it true that determining the greedy defining number of a Latin square is an $\mathcal{NP}$-complete problem?*

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