

Grundy chromatic number of the complement of bipartite graphs

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Abstract

A Grundy k -coloring of a graph G , is a vertex k -coloring of G such that for each two colors i and j with $i < j$, every vertex of G colored by j has a neighbor with color i . The Grundy chromatic number $\Gamma(G)$, is the largest integer k for which there exists a Grundy k -coloring for G . In this note we first give an interpretation of $\Gamma(G)$ in terms of the total graph of G , when G is the complement of a bipartite graph. Then we prove that determining the Grundy number of the complement of bipartite graphs is an NP-Complete problem.

Keywords: Graph coloring, NP-Complete, total graph, edge dominating set.

1 Introduction and preliminaries

In this paper we consider undirected graphs without loops and multiple edges. By a k -coloring of a graph G we mean a proper vertex coloring of G with colors $1, 2, \dots, k$. A Grundy k -coloring of G is a k -coloring of G such that for each two colors i and j with $i < j$, every vertex of G colored by j has a neighbor with color i . The Grundy chromatic number (or simply Grundy number) $\Gamma(G)$, is the largest integer k for which there exists a Grundy k -coloring for G .

The Grundy number of graphs was perhaps introduced for the first time by Christen and Selkow [2]. In [3] another interpretation of Grundy number has been obtained.

Also in [7] the authors studied Grundy number of hypercubes and determined the exact values. Further results on Grundy numbers have been found in [11].

From computational point of view, in [6] a linear algorithm for determining $\Gamma(T)$ has been given, where T is a tree. Also in [9] a polynomial time algorithm for computing Grundy number of partial k -trees has been found. In the unpublished manuscript [4] the NP-Completeness of determining Grundy number of general graphs has been proved. Therefore they gave an affirmative answer to the problem 10.4 posed in the book [8]. In [11] it is proved that the problem of determining the Grundy number of graphs is fixed-parameter tractable.

In the sequel by GRUNDY NUMBER we mean the following decision problem:

GRUNDY NUMBER:

Instance: A graph G and an integer k .

Question: Is $\Gamma(G) \geq k$?

In this paper our aim is to study Grundy number of the complement of bipartite graphs and give a description of it in terms of total graphs. Finally we will prove the NP-Completeness of Grundy number for this restricted class of graphs. Suppose G is the complement of a bipartite graph with a bipartition (X, Y) . By an *extended clique* of G we mean a subgraph of G , which can be defined inductively as follows: An ordinary clique of G is also an extended clique of G . Then, a subgraph H is an extended clique if there exist two non-adjacent vertices $u \in X$ and $v \in Y$ in H such that $H \setminus \{u, v\}$ is an extended clique of G . An extended clique in G , in fact introduces an independent subset I of vertices in the bipartite graph G^c , and a matching in $G^c \setminus I$ with m vertices. By the size of this extended clique we mean the number $|I| + m$.

In this note the concept of *total graph* has been used. The total graph $T(G)$ of a graph G , is the graph whose vertices correspond to the vertices and edges of G , and where two vertices are joined if and only if the corresponding vertices or edges of G are adjacent. The total graph $T(G)$ of G has the following property. Suppose a property \mathcal{P} for a graph G is defined on the set $V(G) \cup E(G)$. Then this property can be converted into a property \mathcal{P}' only on the vertices of $T(G)$. Total graphs were defined for the first time in [1]. We need also the concept of an *edge dominating set*. In a graph G a subset D of edges in G is called an edge dominating set if each edge in $E(G) \setminus D$ has a common end point with an edge in D . By the size of D we simply mean the number of edges in D . In our NP-Completeness result we have transformed the following result of Yannakakis and Gavril [10] concerning the

minimum size of an edge dominating set in any graph G .

Theorem A. *Determining the minimum size of an edge dominating set in a bipartite graph with maximum degree at most 3, is an NP-Complete problem.*

2 Results

An extended clique of G with size t in fact introduces a Grundy coloring of G with at least t colors. The following theorem shows that the converse is also true.

Theorem 1. *Let G be the complement of a bipartite graph. Then there exists an extended clique in G with size $\Gamma(G)$.*

Proof. Suppose that $\Gamma(G) = t$. We prove by induction on t that an extended clique in G with size t can be obtained by a Grundy coloring with t colors in G . We may suppose that the class of vertices colored by 1 in a Grundy t -coloring of G consists of two vertices. Otherwise, if any color class consists of one vertex then clearly $\omega(G) = t$, where $\omega(G)$ is the size of a maximum clique in G , and in this case G should be a complete graph for which the theorem holds. Therefore suppose that a color class say the class i contains two vertices and let i be the minimum color having this property. Then by changing the classes 1 and i with each other, we will have a Grundy t -coloring of G with the desired property. Now if we consider two vertices of color 1 and delete them, then the resulting graph H has Grundy number $t - 1$. By induction on t , we may suppose that the Grundy number $t - 1$ for H is obtained by an extended clique in H having size $t - 1$. Now by adding two vertices colored by 1 to that extended clique in H , we obtain an extended clique with size t in G , as required. \square

The theorem in fact shows that if G is the complement of a bipartite graph then $\Gamma(G)$ equals to the maximum size of an extended clique in G .

In the following theorem, by $\alpha(G)$ we mean the maximum number of independent vertices in a graph G .

Theorem 2. *Suppose G is a bipartite graph. Then*

$$\Gamma(G^c) = \alpha(T(G)).$$

Proof. The Grundy number of G^c is the maximum size of an extended clique in G^c and this later size can be taken as the maximum size of a set K of vertices

and edges in G where both vertices and edges are independent and no edge in K is adjacent to a vertex in K . But such a set K in G introduces an independent subset in $T(G)$, and vice versa any independent set in $T(G)$ provides an extended clique in G^c . Therefore $\Gamma(G^c)$ is the same as the maximum number of independent vertices in $T(G)$, namely $\alpha(T(G))$. \square

Theorem 3. *Let G be the complement of a bipartite graph. Then $\Gamma(G) \leq \frac{3\omega(G)}{2}$.*

Proof. The Grundy number $\Gamma(G)$ is the size of some extended clique in G . The later size is of the form $m + |I|$, where m is the size of a matching M in G^c and I an independent subset of vertices in G^c which are not incident to any edge in M . Let $|I| = a + b$, $a \leq b$, where a is the number of vertices in I belonging to one unique partite in G^c . Let us denote the independence number of G^c by α . It is clear that $a \leq \frac{\alpha}{2}$ and $b + m \leq \alpha$. Consequently

$$\Gamma(G) = a + b + m \leq \alpha + a \leq \frac{3\alpha}{2} = \frac{3\omega(G)}{2}.$$

\square

The above theorem implies the existence of a good polynomial time approximation algorithm for Grundy number with a performance ratio $3/2$.

Before we state the next theorem, we mention the following fact from edge dominating sets in bipartite graphs. Let D be an edge dominating set in a bipartite graph, then there is a matching M which is also an edge dominating set and $|M| \leq |D|$. This fact can be easily proved but we omit mentioning its proof here, and refer the reader to the book [5].

Theorem 4. *The following decision problem is NP-Complete:*

Instance: *A bipartite graph G with $\Delta(G) \leq 3$ and an integer k .*

Question: *Is $\Gamma(G^c) \geq k$?*

Proof. A proper vertex coloring of G^c can be easily checked whether it is a Grundy coloring with at least k number of colors. This shows the membership in NP. We transform EDGE DOMINATING SET (*EDS*) problem to our problem. Suppose (G, k) is an instance for *EDS*. We show that $(G^c, n - k)$ is an appropriate instance for GRUNDY NUMBER, where n is the order of G . Let D be an edge dominating set in G with at most k edges. Then there exists a matching M which is also an edge dominating set and $|M| \leq |D|$. Suppose that $M = \{v_1u_1, v_2u_2, \dots, v_mu_m\}$. Let V_0 be those vertices in G which are not incident with any edge in M . Now it turns out that V_0 in conjunction with the pairs $\{v_i, u_i\}, 1 \leq i \leq m$, forms an extended clique in G^c . Its size is $n - m$. Now if $m \leq k$, then $n - m \geq n - k$. Similarly in the reverse procedure, an extended clique in G^c of size m introduces an edge dominating set in G of size $n - m$. This completes the proof. \square

References

- [1] M. Behzad, Graphs and their chromatic numbers, Ph.D. dissertation, Michigan State University, Department of Mathematics, 1965.
- [2] C.A. Christen and S.M. Selkow, Some perfect coloring properties of graphs, *J. Combin. Theory Ser. B*, **27** (1979), 49–59.
- [3] P. Erdos, W.R. Hare, S.T. Hedetniemi, and R. Laskar, On the equality of Grundy and chromatic number of graphs, *J. Graph Theory* **11** (1987), 157–159.
- [4] N. Goyal and S. Vishvanathan, NP-completeness of undirected Grundy numbering and related problems, Manuscript, Bombay 1997.
- [5] R. Häggkvist, A.S. Asratian, T.M.J. Denley, Bipartite graphs and their applications, Cambridge University Press, Cambridge, 1998.
- [6] S.M. Hedetniemi, S.T. Hedetniemi, and T. Beyer, A linear algorithm for the Grundy (coloring) number of a tree, *Congr. Num.*, **36** (1982), 351-363.
- [7] D.G. Hoffman and P.D. Johnson Jr., Greedy colorings and the Grundy chromatic number of the n -cube, *Bulletin of the ICA* **26** (1999), 49–57.
- [8] T.R. Jensen and B. Toft, Graph coloring problems, John Wiley & Sons, 1995.
- [9] J.A. Telle and A. Proskurowski, Algorithms for vertex partitioning problems on partial k -trees, *SIAM J. Discrete Math.*, **10** (1997), 529-550.
- [10] M. Yannakakis and F. Gavril, Edge dominating sets in graphs, *SIAM J. Appl. Math.*, **38** (1980), 364-372.
- [11] M. Zaker, Results on the Grundy chromatic number of graphs, Manuscript, 2003.